Riesz Representation Thm.

Consider Locally Compact Hausdorff (LCH) spaces X and Borel σ-algebra $\mathcal{B}_X$

Recall. LCH $\Leftrightarrow$ $\forall x \in X$ has an open nbhd U s.t. $\overline{U}$ is compact.

Will be concerned w/ Radon measures, which form a class of measures that in some sense try to imitate main prop's of the Lebesgue measure on $\mathbb{R}^n$.

Some definitions: Let μ be a Borel measure on X

① μ is inner regular on $E \in \mathcal{B}_X$ if
$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$$

② μ is outer regular on $E \in \mathcal{B}_X$ if
$$\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ open}\}$$

(3) μ is <u>regular</u> if both inner and outer regular on all $E \in \mathcal{B}_X$

(4) μ is <u>Radon</u> if

(i) $\mu(K) < \infty$ $\forall K \subseteq X$ compact.

(ii) μ is outer regular on all $E \in \mathcal{B}_X$.

(iii) μ is inner regular on open sets.

We will adopt Follands notation. If U is an open set

$$f \prec U$$

space of cont. fcns w/ compact supp

means that $f \in C_c(X)$, $0 \le f \le 1$, and $\operatorname{supp} f \subseteq U$.

Riesz Representation Thm (Basic).

If I is positive[(*)] linear functional on $\mathcal{C}_c(X)$, then $\exists$ unique Radon measure μ s.t.

$$I(f) = \int_X f \, d\mu \, , \quad \forall f \in \mathcal{C}_c(X).$$

Addendum: The Radon measure μ above satisfies:

- $\mu(U) = \sup\{I(f) : f \prec U\}, \ \forall \text{open } U \quad (1)$
- $\mu(K) = \inf\{I(f) : f \geq \chi_K\}, \ \forall \text{cpct } K \quad (2)$

char. fcn of K

(*) I positive means the obvious thing:

$$I(f) \geq 0, \ \forall f \in \mathcal{C}_c(X), \ f \geq 0.$$

Outline of pf:

(a) Establish uniqueness by showing (1)

(b) Define outer measure μ^* starting w/ (1) as def. of candidate for μ.

(c) Show all open U are μ^*-measurable ($\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \cap U^c), \forall E$)

(d) Establish (2) for resulting Borel measure $\mu \Rightarrow \mu$ is Radon.

(e) Finally, show $I(f) = \int f\, d\mu$, $\forall f \in C_c(X)$.

Pf.

(a) Suppose μ is Radon measure as in RRT, i.e. $I(f) = \int f\, d\mu$, $\forall f \in C_c(X)$. Let U be open set. If $f \prec U$ then clearly $\int f\, d\mu \leq \mu(U)$.

Suppose $c < \mu(U)$. Since μ is inner reg. on open sets $\exists\, K \subseteq U$ cpct s.t. $c < \mu(K) \leq \mu(U)$. By Urysohn (Prop 4.32) $\exists\, f \prec U$ s.t. $f = 1$ on K. Thus

$$I(f) = \int f\, d\mu \geq \mu(K) > c$$

This proves (1). Thus, μ is uniquely determined on open sets. Since μ is outer regular on $\mathcal{B}_X$,

$$\left(\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ open}\} \right)$$

we conclude that μ is uniquely det. on $\mathcal{B}_X$, which establishes uniqueness.

(b) To establish existence, we define a candidate by (1), and define

$$\mu^*(E) = \inf\{\mu(U) : E \subseteq U\}, \quad E \subseteq X \quad (3)$$

Lemma 1. μ^* is an outer measure.

Pf of L1. By def of μ^* and Prop 1.10, suffices to show that for any seq. $\{U_j\}$ of open sets

$$\mu(U) \leq \sum_j \mu(U_j); \quad U = \bigcup_j U_j.$$

Pick $\varepsilon > 0$. By def $\exists f \prec U$ s.t.

$$I(f) \geq \mu(U) - \varepsilon.$$

Let $K = \mathrm{supp}(f) \subseteq U$ cpct. Thus, $\exists$ n s.t. $K \subseteq \bigcup_{j=1}^{n} U_j$.

Prop 4.41

There is a partition of unity $\{g_j\}_{j=1}^{n}$ subordinate to $\{U_j\}_{j=1}^{n}$, i.e.,

- $g_j \prec U_j$
- $\sum_{j=1}^{n} g_j = 1$

Consider $f_j = f g_j \prec U_j$. Note:

$$f = \sum_{j=1}^{n} f_j .$$

$\Rightarrow$

$$\mu(U) \le I(f) + \varepsilon = \sum_{j=1}^{n} I(f_j) + \varepsilon$$

$$\le \sum_{j=1}^{n} \mu(U_j) + \varepsilon \le \sum_{j=1}^{\infty} \mu(U_j) + \varepsilon.$$

Since $\varepsilon > 0$ arbitrary, we have proved the lemma. $\blacksquare$

(C) <u>Lemma 2</u>. All open sets are μ^*-measurable.

<u>Pf.</u> Let $E \subseteq X$ and U open. Suffices to show

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \cap U^c). \quad (4)$$

Pick $\varepsilon > 0$ and V open s.t. $\mu^*(E) \ge \mu(V) - \varepsilon$

Clearly, for open V $\mu^*(V) = \mu(V)$, and

$V \cap U^c = V \setminus U$.

Choose $f_1 \prec V\cap U$ s.t. $I(f_1) \geq \mu(V\cap U) - \varepsilon$
and then $f_2 \prec V\setminus \operatorname{supp} f_1 \supseteq V\setminus U$ s.t.

$$I(f_2) \geq \mu(V\setminus \operatorname{supp} f_1) - \varepsilon$$

Observe that $f = f_1 + f_2 \prec V$ and
$\mu(V\setminus \operatorname{supp} f_1) \geq \mu^*(V\setminus U) \Rightarrow$

$$\mu^*(E) \geq \mu(V) - \varepsilon \geq I(f) - \varepsilon = I(f_1) + I(f_2) - \varepsilon$$

$$\geq \underset{\parallel}{\mu(V\cap U)} + \mu(V\setminus \operatorname{supp} f_1) - 3\varepsilon$$

$$\geq \mu^*(V\cap U) + \mu^*(V\setminus U) - 3\varepsilon$$

$$\geq \mu^*(E\cap U) + \mu^*(E\setminus U) - 3\varepsilon.$$

Since $\varepsilon > 0$ arbitrary, we have proved (4) and, hence, Lemma 2. ☒

By Caratheodory's Thm, we get a <u>Borel measure</u> μ (actually, can extend to a complete measure).

(The fact that μ now denotes the resulting measure should cause no confusion.)

<u>Lemma 3</u>. (2) holds for the Borel measure μ.

(d) <u>Pf.</u> Since I positive, $0 \le f \le g \Rightarrow$
$I(f) \le I(g)$.

Again, pick $\varepsilon > 0$ and $f \geq \chi_K$ for some cpct K. Consider open set $U_\varepsilon = \{x : f > 1-\varepsilon\} \Rightarrow$ $K \subset U_\varepsilon$. $\Rightarrow \mu(K) \le \mu(U_\varepsilon) = \sup\{I(g) : g \prec U_\varepsilon\}$.

If $g \prec U_\varepsilon$, then $g \le \frac{f}{1-\varepsilon} \Rightarrow I(g) \le I\left(\frac{f}{1-\varepsilon}\right)$.

$$\Rightarrow \mu(K) \le \frac{1}{1-\varepsilon} I(f) \Rightarrow \{\varepsilon \to 0\}$$

$$\mu(K) \le \inf\{I(f) : f \geq \chi_K\}.$$

By outer reg. of μ (as noted, follows from def. of μ^*), $\exists$ open $U \supseteq K$ s.t. $\mu(K) \geq \mu(U) - \varepsilon$

By Urysohn, $\exists f \geq \chi_K$, $f \prec U$. Thus,
$\mu(K) \geq \mu(U) - \varepsilon \geq I(f) - \varepsilon$. $\Rightarrow \{\varepsilon \to 0\}$

$\mu(K) \geq \inf\{I(f) : f \geq \chi_K\}$ as desired. ∎

We now have both (1) and (2) for the Borel measure μ. We claim that μ is Radon: Need to check

- $\mu(K) < \infty$, $\forall$ cpct K. (Clear from (2))
- outer regular on $\mathcal{B}_X$ (as noted, this follows by def. of μ^*).
- inner reg. on open sets.

Let's prove last •: Let U open, pick $\varepsilon > 0$ as usual. By (1), $\exists$ $f \prec U$ s.t. $I(f) \geq \mu(U) - \varepsilon$. Let $K = \operatorname{supp} f$. For any $g \geq \chi_K$ we have $g \geq f \Rightarrow$ $I(g) \geq I(f) \geq \mu(U) - \varepsilon$. By (2) $\exists$ $g_\varepsilon \geq \chi_K$ s.t. $I(g_\varepsilon) \leq \mu(K) + \varepsilon \Rightarrow$

$$\mu(U) \leq I(g_\varepsilon) + \varepsilon \leq \mu(K) + 2\varepsilon$$

Since $\varepsilon > 0$ arbitrary, we find that

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ cpct}\}.$$

$\blacksquare$

(c) We now have a candidate Radon measure μ_I.
Last, but not least, we must show:

<u>Lemma 4</u> $I(f) = \int f \, d\mu$, $\forall f \in C_c(X)$ (5)

Pf. Since any $f \in C_c(X)$ decomposes into real & imag. parts and each of these decomposes into a pos. and neg. part, Linearity of the integral $\Rightarrow$ suffices to check (5) for $0 \le f \le 1$. Pick such f.

<u>Construction</u>: Let $K_0 = \text{supp } f$. Choose $n \in \mathbb{Z}_+$ and define $K_j = \{x : f \geq j/n\}$, $1 \le j \le n$. $\Rightarrow$

all compact

$K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots \supseteq K_n$ (where K_j could be $\emptyset$.)

Let $$f_j(x) = \begin{cases} \frac{1}{n}, & x \in K_j \\ f(x) - \frac{j-1}{n}, & x \in K_{j-1} \setminus K_j \\ 0, & x \in K_{j-1}^c \end{cases}$$

One checks readily $f_j \in \mathcal{C}_c(X, [0,1])$ and

$$\frac{1}{n}\chi_{K_j} \le f_j \le \frac{1}{n}\chi_{K_{j-1}}. \qquad (6)$$

if $x \in K_{j-1} \setminus K_j$ then $f < j/n \Rightarrow f_j < j/n - \frac{j-1}{n} = \frac{1}{n}$

Also note that if $x \in K_{\ell-1} \setminus K_\ell$, for any $1 \le \ell \le n$,

$$f_j(x) = 0, \quad j \ge \ell+1, \Rightarrow$$

$$\sum_{j=1}^{n} f_j(x) = \sum_{j=1}^{\ell} f_j(x) = \sum_{j=1}^{\ell-1} \frac{1}{n} + f(x) - \frac{\ell-1}{n} = f(x)$$

$$\Rightarrow \sum_{j=1}^{n} f_j = f.$$

Integrate (6) wrt $d\mu$ and sum over $j \Rightarrow$

$$\frac{1}{n}\sum_{j=1}^{n} \mu(K_j) \le \int f\,d\mu \le \frac{1}{n}\sum_{j=0}^{n-1} \mu(K_j) \qquad (7)$$

Next, the first inequal. in (6) and (2) $\Rightarrow$

$$\mu(K_j) \leq n\,I(f_j).$$

Since μ is outer reg. on all Borel sets, for any $\varepsilon > 0$ we can find open $U \supseteq K_{j-1}$ s.t. $\mu(K_{j-1}) \geq \mu(U) - \varepsilon$. Note that $nf_j \prec U \Rightarrow$ (by (1)); $nI(f_j) \leq \mu(U) \leq \mu(K_{j-1}) + \varepsilon$ $\Rightarrow \{\varepsilon \to 0\} \Rightarrow nI(f_j) \leq \mu(K_{j-1})$. Again, summing over j and dividing by n $\Rightarrow$

$$\frac{1}{n}\sum_{j=1}^{n} \mu(K_j) \leq I(f) \leq \frac{1}{n}\sum_{j=0}^{n-1} \mu(K_j). \quad (8)$$

(7) + (8) $\Rightarrow$ {consider $I(f) - \int f\,d\mu$ & $\int f\,d\mu - I(f)$}

$$\left| I(f) - \int f\,d\mu \right| \leq \frac{\mu(K_0) - \mu(K_n)}{n} \leq \frac{\mu(K_0)}{n}$$

Since $K_0 \supseteq$ supp f and indep. of n cpct & $\mu(K_0) < \infty$, letting $n \to \infty$ we find $I(f) = \int f\,d\mu$ as desired. which proves Lemma 4 ▨ This completes pf of RRT. ▨

Rmk. The measure $\bar{\mu}$ we obtain (by Caratheodory) by restricting the outer measure μ^* in the proof to the full σ-algebra $\mathcal{M}$ of all μ^*-measurable sets (which as we proved contains $\mathcal{B}_X$) is a complete measure, which turns out to be the completion of the Radon measure μ if μ is σ-finite and the saturation of the completion of μ in the general case.